

Predictable Forward Performance Processes: The Binomial Case

Bahman Angoshtari* Thaleia Zariphopoulou† Xun Yu Zhou‡

November 15, 2016

Abstract

We introduce a new class of forward performance processes that are predictable with regards to an underlying filtration and are updated in discrete time. Such performance criteria may accommodate short-term predictability of asset returns, sequential learning and other dynamically unfolding factors affecting optimal portfolio choice. We analyze in detail a binomial model. We show that the key step is to solve a single-period inverse investment problem, which we study in detail. In particular, We reduce this inverse problem to an iterative (i.e. single variable) functional equation and establish conditions for existence and uniqueness of solutions in the class of inverse marginal functions. This functional equation is the counterpart of the stochastic partial differential equation that characterizes the continuous-time forward performance processes.

Keywords: Optimal investment, forward performance processes, binomial model, inverse investment problem, iterative functional equation.

1 Introduction

This paper contributes to the performance valuation of portfolio strategies under forward investment performance criteria. These criteria are modeled as a stochastic process, $U(x, t)$, which propagates “forward in time” in a time-consistent manner as market evolves. Specifically, $U(x, t)$ is a forward performance process, with respect to an underlying filtration (\mathcal{F}_t) , if for any admissible wealth process, say X_t^π , with π_t being an investment strategy, the process $U(X_t^\pi, t)$ is a supermartingale, and there exists a strategy π_t^* for which $U(X_t^*, t)$ becomes a martingale. In essence, the forward performance process is “self-generated” by ensuring that the Dynamic Programming Principle holds for any arbitrary trading period. The value function process in the classical expected utility paradigm is an example of a forward performance, if one requires that at the end of the horizon T , $U(x, T)$ is a given deterministic $(\mathcal{F}_0$ -mble) function.

*Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA. Email: bango@umich.edu.

†Department of Mathematics and IROM, The University of Texas at Austin, Austin and the Oxford-Man Institute, University of Oxford, Oxford, UK. Email: zariphop@math.utexas.edu.

‡Department of IEO, Columbia University, New York, NY 10027. Email: xz2574@columbia.edu

The motivation to develop forward performance criteria is threefold. Firstly, it is frequently difficult to assess and specify the utility function in a (possibly remote) future time. It is more plausible to know the utility, or the preferred allocations, for the immediate future or for the upcoming period (see, for example, [2] and [12]). Secondly, in the classical expected utility setting, one needs to pre-commit to a market model for the entire investment period, say $[0, T]$. Even if one allows for model uncertainty, the choice of the set of possible models is still biding [5]. More generally, uncertainty might be generated by both asset evolution and exogenous factors, and it is quite difficult to model them accurately especially for a long time ahead (see [13]). Thirdly, it is often the case, for example in risk measures, indifference valuation and in investment with rolling horizons, that horizon flexibility is desired, a feature not present in the classical setting (see, among others, [7] and [16]).

Forward investment performance criteria were introduced by one of the authors and Musiela in [9] and [10], and we refer the reader to [11], [12], [3] for further details. With the exception of the discrete exponential case, studied in [9] and [8], in the context of exponential indifference prices, the existing results have so far focused exclusively on continuous-time settings in Ito-diffusion markets, in which both trading and performance valuation are done continuously in time.

In [11] it was shown that the forward performance process is associated with an ill-posed stochastic PDE which has been subsequently studied in [3], [14], [13] and [15]. Despite the technical challenges that this forward performance SPDE presents (ill-posedness, degeneracies and volatility specification) the continuous-time cases are tractable because stochastic calculus can be employed and infinitesimal arguments are, in turn, developed. However, continuous-time settings might not be adequately useful for applications. Indeed, in investment practice, trading occurs at discrete times and not continuously.

More importantly, performance criteria are directly or indirectly determined by individuals, such as higher-level managers or clients, that are different from the portfolio manager. These “performance evaluators” use information sets that are different, both in terms of contents and updating frequency, from the ones used by the portfolio manager. For example, a portfolio manager may have access to various data sets, proprietary forecasting models, and sophisticated trading strategies which are out of reach of (or simply deemed as “too detailed” to be considered by) the performance evaluator. Similarly, even if trading can happen at extremely higher frequencies (hence almost close to continuous trading), performance assessment takes place at a much slower pace, e.g. a senior manager will not keep track of the performance of a portfolio as frequently as the subordinate portfolio manager in charge of that portfolio.

The aim herein is to initiate the development and a systematic study of investment performance processes that are *discrete* in time, while trading can be either discrete or continuous. Within this class, we first introduce forward performance processes that are *predictable* with regards to the information at the most recent assessment time. Namely, if $0 = t_0, t_1, \dots, t_n, \dots$ are the performance evaluation times, then we require that, for $n = 1, 2, \dots$, $U(x, t_n)$ is \mathcal{F}_{n-1} -mble.

We are motivated to introduce this class of criteria for two reasons. Firstly, it is natural to define at the beginning of the assessment period the criterion we use for the end of it. This is, for example, the case in the expected utility framework where there is only one evaluation period and the terminal utility is deterministic (\mathcal{F}_0 -mble).

Secondly, it is more feasible to estimate the market parameters for just one evaluation

period ahead, than for longer periods. For example, the volatility can be reliably forecasted for a short time ahead using the so-called realized volatility introduced by the seminal work [1]. Similarly, short-term predictability of equity risk premium has been subject to extended studies in the last three decades (see, among others, [4]) and is consistent with how active investors such as hedge funds operate.

To highlight the key ideas of predictable forward investment processes we start our analysis with a simple but still rich enough setting. The market consists of two securities, a riskless asset and a stock whose price evolves according to a binomial model at times $0 = t_0, t_1, \dots, t_n, \dots$, at which the forward evaluation also occurs. The market model is more general than the standard one, in that the asset return, say R_{n+1} , during $[t_n, t_{n+1})$ takes values R_{n+1}^u, R_{n+1}^d that are \mathcal{F}_n -mble, with “probability” p_{n+1} , which is also \mathcal{F}_n -mble. This setting allows for updating of returns and their probabilities, as the market unfolds from one evaluation period to the next.

As we show in Section 4, the investor starts at $t = 0$ and chooses (i.e. estimates) the market parameters (R_1^u, R_1^d, p_1) for the upcoming trading/evaluation period $[0, t_1]$. She also chooses her initial deterministic utility, say $U_0(x)$, and seeks a utility (performance) function $U_1(x)$ also deterministic, since $U_1(x) \in \mathcal{F}_0$, such that the pair (U_0, U_1) is consistent with the investment problem in $[0, t_1]$, with U_0, U_1 being respectively the value function and terminal utility.

In turn, at t_1 , the agent observes (R_2^u, R_2^d, p_2) for the next period $[t_1, t_2]$. She then solves a similar “inverse investment problem” for a pair (U_1, U_2) , with $U_2 \in \mathcal{F}_1$ to be determined, starting with initial wealth $X_1^*(x) \in \mathcal{F}_1$, generated at the end of the previous period. Proceeding iteratively *forward in time*, a predictable investment criterion is constructed together with the optimal allocations and their wealths.

The underlying optimization problem then is a *single-period inverse investment* problem, which needs to be solved “period-by-period” conditionally on the information at the beginning of the period. We analyze this problem in detail. The key equation turns out to be a linear functional equation, which relates the inverse marginal processes, at the beginning and the end of each trading period, with coefficients depending on market inputs, see (9). We analyze this equation in detail, and establish conditions for existence and uniqueness of solutions in the class of inverse marginal functions, see Theorems 8 and 10.

The paper is structured as follows. In Section 2, we introduce the notion of predictable forward performance processes in a general market setting. We consider a binomial model with random parameters in Section 3. In Section 4, we apply the definition of predictable forward performance processes to the binomial model, and show that their construction reduces to solving an inverse investment problem. In Section 5, the inverse problem is shown to be equivalent to an iterative (i.e. single variable) functional equation. Sufficient existence and uniqueness condition as well as the explicit solution to the functional equation are derived in Section 6. We conclude in section 7. Proofs of the main theorems are included in the Appendix.

2 Predictable forward performance processes

In this section, we propose the definition of predictable forward performance processes in a rather general market model. In the next sections, we restrict the market setting to

a binomial model and provide a detail discussion on existence and construction of these performance processes.

The investment paradigm is cast in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ augmented with a filtration (\mathcal{F}_t) , $t \geq 0$. We denote by $\mathcal{X}(t, x)$ the set of the associated admissible wealth processes X_s , $s \geq t$, starting with $X_t = x$ and such that X_s is \mathcal{F}_s -mble. The term “admissible” is for now generic and will be refined once a specific market model is introduced in the next section.

We call a function $U : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a *utility (or performance) function* if $U \in C^2(\mathbb{R}^+)$, $U' > 0$, $U'' < 0$, and satisfies the Inada conditions, $\lim_{x \downarrow 0} U'(x) = \infty$ and $\lim_{x \uparrow \infty} U'(x) = 0$.

For any σ -algebra $\mathcal{G} \subseteq \mathcal{F}$, the set of \mathcal{G} -measurable *utility (performance) functions* is defined as

$$\mathcal{U}(\mathcal{G}) = \{U : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R} \mid U(x, \cdot) \text{ is } \mathcal{G}\text{-mble for each } x \in \mathbb{R}^+, \\ \text{and } U(\cdot, \omega) \text{ is a utility function a.s.}\}$$

In other words, the elements of $\mathcal{U}(\mathcal{G})$ are entirely predicted based on the information contained in \mathcal{G} .

Next, we introduce the concept of predictable forward performance processes. To ease the notation, we skip the ω -argument throughout.

Definition 1. *Let discrete time points $0 = t_0 < t_1 < \dots < t_n < \dots$ be given and (\mathcal{F}_t) , $t \geq 0$, be an underlying filtration. A family of random functions $\{U_0, U_1, U_2, \dots\}$ is a predictable forward performance process with respect to (\mathcal{F}_t) if, for $n = 1, 2, \dots$, $\mathcal{F}_n = \mathcal{F}_{t_n}$ and $X_{t_n} = X_n$,*

(i) U_0 is a deterministic utility function and $U_n \in \mathcal{U}(\mathcal{F}_{n-1})$.

(ii) For any initial wealth $x > 0$ and any admissible wealth process $X_t \in \mathcal{X}(0, x)$,

$$U_{n-1}(X_{n-1}) \geq E_{\mathbb{P}}[U_n(X_n) \mid \mathcal{F}_{n-1}].$$

(iii) For any initial wealth $x > 0$, there exists an admissible wealth process $X_t^* \in \mathcal{X}(0, x)$ such that

$$U_{n-1}(X_{n-1}^*) = E_{\mathbb{P}}[U_n(X_n^*) \mid \mathcal{F}_{n-1}].$$

We stress that, as we discussed in the introduction, there are no specific assumptions on the market model and how often trading occurs. The market model can be discrete or continuous, and for the latter case, the trading can be discrete or continuous. Furthermore, if trading is discrete, the rebalancing periods do not need to be aligned with the performance assessment times.

In practice, as mentioned in the Introduction, it is typically the case that trading occurs more frequently than performance evaluation, but the above definition accommodates cases when there is perfect alignment - as in the binomial model herein - and the less realistic case when trading lags.

The simplest case of a discrete forward performance process was introduced in [9] for a binomial model in the context of forward exponential indifference prices. For the special case of time-monotone processes, the solution turned out to be predictable even though the predictability concept per se was not considered therein.

The general definition of forward performance processes was proposed for Ito-diffusion markets in [10] (see, also [11] and [12], and for variations of the definition, see [3], [13] and [17]). Definition 1 is therefore a discrete analogue of it with the fundamental element being condition (i), which explicitly requires that the performance function at the *next* upcoming assessment time be *completely determined* from the information up to the *present* time.

As in the continuous-time case, properties (ii)-(iii) draw from Bellman's principle of optimality, which stipulates that the processes $U_n(X_n)$ and $U_n(X_n^*)$, $n = 0, 1, \dots$, are respectively a supermartingale and a martingale with respect to the filtration (\mathcal{F}_n) . Since Bellman's principle underlines time consistency, properties (ii)-(iii) ensure that the investment problem is time consistent under the predictable forward performance criterion.

We also note that predictability of risk preferences is implicitly present in the classical expected utility in a finite horizon settings, say $[0, T]$, in which a deterministic utility for T is pre-chosen at initial time $t_0 = 0$, and it is thus \mathcal{F}_0 -mble.

Definition 1 leads to a general scheme for constructing predictable forward performance functions. Indeed, starting from an initial datum U_0 , given at time $t_0 = 0$, the entire family U_1, \dots, U_n, \dots , can be obtained by determining U_n from U_{n-1} iteratively, $n = 1, 2, \dots$, as follows.

Properties (ii)-(iii) dictate that, for each trading period $[t_{n-1}, t_n]$, we have

$$U_{n-1}(X_{n-1}^*) = \operatorname{ess\,sup}_{X_n \in \mathcal{X}(t_{n-1}, X_{n-1}^*)} E_{\mathbb{P}}[U_n(X_n) | \mathcal{F}_{n-1}]. \quad (1)$$

At instant t_{n-1} , since \mathcal{F}_{n-1} is realized, the random functions U_{n-1} and U_n are both deterministic and so is X_{n-1}^* . This, in turn, suggests that we should consider the following “single-period” investment problem

$$U_{n-1}(x) = \operatorname{ess\,sup}_{X_n \in \mathcal{X}_{n-1,n}(x)} E_{\mathbb{P}}[U_n(X_n) | \mathcal{F}_{n-1}], \quad (2)$$

for $x > 0$, where, with a slight abuse of notation, we use $\mathcal{X}_{n-1,n}(x)$ to denote the set of admissible wealths at t_n starting at t_{n-1} with wealth x .

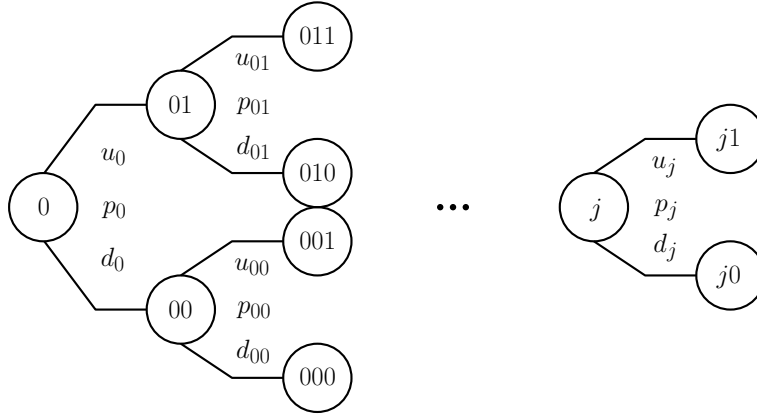
Therefore, if we are able to determine, for each $n = 1, 2, \dots$, a performance function $U_n \in \mathcal{U}(\mathcal{F}_{n-1})$, with $U_{n-1} \in \mathcal{U}(\mathcal{F}_{n-2})$ and (U_n, U_{n+1}) satisfying (2), then we will have an iterative scheme to construct the *entire* predictable forward performance, starting from U_0 .

One readily recognizes that (2) would be the classical expected utility problem if the objective were to derive U_{n-1} from U_n , with U_n being a deterministic utility function. Therefore, what we consider now is an *inverse investment* problem in that we are given its initial value function and we seek a terminal utility that is consistent with it, with both these functions being deterministic (conditionally on \mathcal{F}_{n-1}).

To our knowledge, such inverse problems have not been considered before. We stress that while the forward performance approach is throughout based on inverse problems of similar nature, the existing continuous-time cases do not satisfy requirement (i) in Definition 1 because $U_t(x)$ is \mathcal{F}_t -mble. Indeed, this is the case for time-monotone and factor-form performance processes given, respectively, by $U_t(x) = u\left(x, \int_0^t |\lambda_s|^2 ds\right)$ and $U_t(x) = V(x, Y_t, t)$, where u and V are deterministic functions, and λ and Y the market price of risk and market factor vector processes (see, [11],[14],[13] and [15]). The only continuous-time predictable case is $u(x, |\lambda|^2 t)$ in log-normal markets, for it is deterministic.

Herein, we start a concise study of predictable performance processes with a binomial market model. As we will see, while this is one of the simplest discrete-time market models, its analysis is not trivial.

Figure 1: The binomial tree with random parameters



3 A binomial market model with random parameters

We consider a market with two traded assets, a riskless bond and a stock. The bond is assumed to offer zero interest rate. The stock prices at times t_0, t_1, \dots , evolve according to a binomial model that we discuss next.

Consider the non-recombinant binomial tree in Figure 1. We use the following notation to index the nodes of the tree. The initial node is indexed by 0. If the index of a node is i , then the index of the node that is reached by an upward (resp. downward) move is $i1$ (resp. $i0$). The set of all indices for nodes of the tree is

$$\mathcal{N} = \{0, 01, 00, 011, 010, 001, 000, \dots\} = \{0i_1 \dots i_n : i_1, \dots, i_n \in \{0, 1\}, n \geq 0\}.$$

In practice, future returns of assets are predicted, for a short time ahead, using various sources such as publicly available data and proprietary qualitative or quantitative information. To be able to use these volatile predictions as an input to our model, we assume that all the parameters of the binomial tree are random variables.

In particular, we assume that p_j, u_j, d_j , for $j \in \mathcal{N}$, and ξ_n , for $n \geq 1$, are random variables in a measurable space (Ω, \mathcal{F}) , such that $0 < p_j < 1$, $0 < d_j < 1 < u_j$ and $\xi_n \in \{0, 1\}$. Below, we provide extra assumptions under which p_j is interpreted as the “conditional probability” of an upward jump if node j is visited; $u_i - 1$ and $d_i - 1$ represent the next-period return of the stock if node j is visited; and $\xi_n \in \{0, 1\}$ indicates whether the stock jumps up or down during the period $[t_{n-1}, t_n]$.

We assume that the values of p_j, u_j, d_j are observed only if node j is visited.¹ Therefore, at node $j = 0i_1i_2 \dots i_n$, we have observed $p_0, u_0, d_0, p_{0i_1}, u_{0i_1}, d_{0i_1}, \dots, p_{0i_1 \dots i_n}, u_{0i_1 \dots i_n}, d_{0i_1 \dots i_n}$. We define the sigma algebra \mathcal{G}_j as the information available at node $j = 0i_1i_2 \dots i_n$, namely,

$$\mathcal{G}_j = \sigma(p_0, u_0, d_0, p_{0i_1}, u_{0i_1}, d_{0i_1}, p_{0i_1i_2}, u_{0i_1i_2}, d_{0i_1i_2}, \dots, p_j, u_j, d_j).$$

For example $\mathcal{G}_0 = \sigma(p_0, u_0, d_0)$, and $\mathcal{G}_{01} = \sigma(p_0, u_0, d_0, p_{01}, u_{01}, d_{01})$.

We assume that the historical measure \mathbb{P} is a probability measure on (Ω, \mathcal{F}) that satisfies the following assumption.

¹For example, one can assume that these values are the outcome of a sequential learning procedure that is conducted when the market is at node j .

Assumption 2. For all $j = 0, 1, \dots, i_n \in \mathcal{N}$:

- (i) $0 < d_j < 1 < u_j$, \mathbb{P} -almost surely; and,
- (ii) $0 < p_j < 1$, $\xi_{n+1} \in \{0, 1\}$, and $p_j = E_{\mathbb{P}}\left(\mathbf{1}_{\{\xi_{n+1}=1\}}|\mathcal{G}_j\right)$, \mathbb{P} -almost surely.

Assumption 2.(i) is the standard no-arbitrage condition. Assumption 2.(ii) is adapted such that p_j represent the probability of an upward jump if the state of the market is j .

We assume no further information about the physical measure. In particular, we do not assume that \mathbb{P} is known. \mathbb{P} can be any measure satisfying Assumption 2.

Next, we define the information available at time t_n , $n \in \{0, 1, \dots\}$, by

$$\mathcal{F}_n = \sigma\left(p_0, u_0, d_0, \xi_1, p_{0\xi_1}, u_{0\xi_1}, d_{0\xi_1}, \dots, \xi_n, p_{0\xi_1 \dots \xi_n}, u_{0\xi_1 \dots \xi_n}, d_{0\xi_1 \dots \xi_n}\right).$$

For example $\mathcal{F}_0 = \sigma(p_0, u_0, d_0)$, and $\mathcal{F}_1 = \sigma(p_0, u_0, d_0, \xi_1, p_{0\xi_1}, u_{0\xi_1}, d_{0\xi_1})$. Note that $\{\mathcal{F}_n\}_{n \geq 0}$ is a filtration, i.e. $\mathcal{F}_{n-1} \subseteq \mathcal{F}_n$.

For $n \geq 1$, the return of the stock over period $[t_{n-1}, t_n]$ is given by

$$R_n = R_n^u \mathbf{1}_{\{\xi_n=1\}} + R_n^d \mathbf{1}_{\{\xi_n=0\}},$$

where

$$R_n^u = u_{0\xi_1 \dots \xi_{n-1}}, \quad \text{and} \quad R_n^d = d_{0\xi_1 \dots \xi_{n-1}}.$$

For future reference, we also define (with slight abuse of notation)

$$p_n = p_{0\xi_1 \dots \xi_{n-1}}; \quad n \geq 1.$$

In particular, R_n is \mathcal{F}_n -mble while p_n , R_n^u and R_n^d are \mathcal{F}_{n-1} -mble.

The investor trades between the stock and the bond using self-financing strategies. She starts at $t_0 = 0$ with total wealth $x > 0$ and rebalances her portfolio at times t_n , $n \geq 0$.

At the beginning of each period, say $[t_n, t_{n+1})$, she chooses the amount π_{n+1} to be invested in the stock (and the rest in the bond) for this period. In turn, her wealth process, denoted by X_n^π , $n \in \{0, 1, \dots\}$, evolves according to the budget constraint

$$X_{n+1}^\pi = X_n^\pi + \pi_{n+1}(R_{n+1} - 1),$$

with $X_0 = x$. The agent is allowed to short the stock but her wealth can never become negative, and thus π_{n+1} must satisfy

$$-\frac{X_n^\pi}{R_{n+1}^u - 1} \leq \pi_{n+1} \leq \frac{X_n^\pi}{1 - R_{n+1}^d}; \quad n \in \{0, 1, \dots\}. \quad (3)$$

We call an investment strategy $\pi = \{\pi_n\}_{n=1}^\infty$ *admissible* if it is self-financing, π_n is \mathcal{F}_{n-1} -mble, and (3) is satisfied \mathbb{P} -almost surely. A wealth process $X = \{X_n^\pi\}_{n=0}^\infty$ is then admissible if the π that generates it is admissible.

We recall that $\mathcal{X}(n, x)$ is the set of admissible wealth processes $\{X_m\}_{m=n}^\infty$, assuming $X_n = x$.

We also introduce the single-step auxiliary set of admissible portfolios π_{n+1} , chosen at t_n for the trading period $[t_n, t_{n+1})$ assuming wealth x at t_n , by

$$\mathcal{A}_{n,n+1}(x) = \left\{ \pi_{n+1} : \pi_{n+1} \text{ is } \mathcal{F}_n\text{-mble, } -\frac{x}{R_{n+1}^u - 1} \leq \pi_{n+1} \leq \frac{x}{1 - R_{n+1}^d} \right\}; \quad x > 0,$$

as well as the corresponding set of admissible wealth processes

$$\mathcal{X}_{n,n+1}(x) = \{x + \pi_{n+1}R_{n+1} : \pi_{n+1} \in \mathcal{A}_{n,n+1}(x)\}; \quad x > 0.$$

4 Problem statement and reduction to the inverse investment Problem

In this section, we consider predictable forward performance process as in Definition 1, and show that their construction reduces to solving an inverse investment problem.

We assume that the agent starts with an initial utility U_0 and evaluates the performance of her investment strategies at times t_1, t_2, \dots introduced above, with associated criteria U_1, U_2, \dots satisfying Definition 1.

We now discuss a procedure how to construct a predictable forward performance process starting from U_0 , and determining U_n from U_{n-1} , iteratively for $n = 1, 2, \dots$.

At $t_0 = 0$, equation (1) becomes

$$U_0(x) = \operatorname{ess\,sup}_{X_1 \in \mathcal{X}(0,x)} E_{\mathbb{P}} \left[U_1(X_1) \middle| \mathcal{F}_0 \right] = \sup_{\pi_1 \in \mathcal{A}_{0,1}(x)} E_{\mathbb{P}} \left[U_1 \left(x + \pi_1(R_1 - 1) \right) \right]; \quad x > 0. \quad (4)$$

Since the market parameters (R_1^u, R_1^d, p_1) and the initial datum U_0 are known, finding a deterministic (\mathcal{F}_0 -mble) U_1 reduces to the single-period inverse investment problem discussed in the previous section. Let us assume that we are able to solve this inverse problem to obtain U_1 .

At $t = t_1$, the investor observes the realization of the stock return R_1 as well as the parameters (R_2^u, R_2^d, p_2) for the second trading period $[t_1, t_2)$. Setting $n = 2$ in (1) then yields

$$U_1(X_1^*(x)) = \operatorname{ess\,sup}_{X_2 \in \mathcal{X}(1, X_1^*(x))} E_{\mathbb{P}} [U_2(X_2) | \mathcal{F}_1], \quad (5)$$

where $X_1^*(x)$ is the optimal wealth generated at t_1 , starting at x at $t_0 = 0$.

From classical results in the traditional expected utility problem, it is well known (see, also, Theorem 4 herein) that $X_1^*(x) = I_1(\rho_1 U_0'(x))$, $x > 0$, where $I_1 = (U_1')^{-1}$ and ρ_1 is the pricing kernel over the period $[0, t_1)$, given by

$$\rho_1 = \frac{1 - R_1^d}{p_1(R_1^u - R_1^d)} \mathbf{1}_{\{\xi_1=1\}} + \frac{R_1^u - 1}{(1 - p_1)(R_1^u - R_1^d)} \mathbf{1}_{\{\xi_1=0\}}.$$

The mapping $x \rightarrow X_1^*(x)$ is strictly increasing for each $x > 0$ and of full range, since I_1 and U_0' are both strictly decreasing functions, $\rho_1 > 0$, and the Inada conditions yield $X_1^*(0) = 0$ and $X_1^*(\infty) = \infty$.

Using that $X_1^*(x)$ is \mathcal{F}_1 -mble and, moreover, that at $t = t_1$, the parameters (R_2^u, R_2^d, p_2) together with U_1 are all known, we easily deduce that (5) reduces, with a slight abuse of notation, to finding $U_2(\cdot) \in \mathcal{U}(\mathcal{F}_1)$ solving, for all $x > 0$,

$$U_1(x) = \operatorname{ess\,sup}_{\pi_2 \in \mathcal{A}_{1,2}(x)} E_{\mathbb{P}} \left[U_2(x + \pi_2(R_2 - 1)) \middle| \mathcal{F}_1 \right],$$

with U_1 given. In other words, one needs to solve yet another inverse investment problem, mathematically identical to (4).

At $t = t_n$, in exactly the same manner as above, we have to solve

$$U_n(x) = \operatorname{ess\,sup}_{\pi_{n+1} \in \mathcal{A}_{n,n+1}(x)} E_{\mathbb{P}} \left[U_{n+1}(x + \pi_{n+1}(R_{n+1} - 1)) \middle| \mathcal{F}_n \right]; \quad \forall x > 0,$$

thereby deriving U_{n+1} from U_n , with $U_{n+1} \in \mathcal{U}(\mathcal{F}_{n+1})$, and with the parameters (R_n^u, R_n^d, p_n) known.

Thus, all terms of the predictable forward performance process can be obtained, starting from any arbitrary initial wealth $x > 0$ and proceeding iteratively solving a “period-by-period” inverse optimization problem. Moreover, as we show in the next section, we also recover the optimal investment and its wealth for each period. Therefore, the key step in the entire construction is to solve the *single-period inverse investment problem*. We do this next.

5 The single-period inverse investment problem

We focus on the analysis of the inverse investment problem (4). To ease the presentation, we introduce a simplified notation. We set $t_0 = 0, t_1 = 1$ and $R_1 = R$ taking values u and d , $u > 1$ and $0 < d < 1$, with probability $0 < p < 1$ and $1 - p$, respectively. We recall the risk neutral probabilities

$$q = \frac{1 - d}{u - d} \quad \text{and} \quad 1 - q = \frac{u - 1}{u - d},$$

and the pricing kernel

$$\rho_1 = \rho^u \mathbf{1}_{\{R=u\}} + \rho^d \mathbf{1}_{\{R=d\}} := \frac{q}{p} \mathbf{1}_{\{R=u\}} + \frac{1 - q}{1 - p} \mathbf{1}_{\{R=d\}}. \quad (6)$$

The agent starts with wealth $X_0 = x > 0$, and invests the amount π in the stock. Her wealth at $t = 1$ is then given by $X = x + \pi(R - 1)$. The no-bankruptcy constraint (3) becomes $\underline{\pi}(x) \leq \pi \leq \bar{\pi}(x)$, with

$$\underline{\pi}(x) = -\frac{x}{u - 1} < 0 \quad \text{and} \quad \bar{\pi}(x) = \frac{x}{1 - d} > 0.$$

We denote the set of admissible portfolios

$$\mathcal{A}(x) = \{\pi \in \mathbb{R}, \text{ and } \underline{\pi}(x) \leq \pi \leq \bar{\pi}(x), \}; \quad x > 0.$$

Given a utility function U_0 , we seek another utility function U_1 such that, for all $x > 0$

$$U_0(x) = \sup_{\pi \in \mathcal{A}(x)} E_{\mathbb{P}} \left[U_1(x + \pi(R - 1)) \right]. \quad (7)$$

Let \mathcal{U} be the set of deterministic utility functions. We introduce the set of *inverse marginal functions* \mathcal{I} ,

$$\mathcal{I} := \left\{ I \in C^1(\mathbb{R}^+) : I' < 0, \lim_{y \rightarrow +\infty} I(y) = 0, \lim_{y \rightarrow 0^+} I(y) = +\infty \right\}. \quad (8)$$

Note that if functions U and I satisfy $I = (U')^{-1}$, then U is a utility function if and only if I is an inverse marginal function.

Assuming for now that a utility function U_1 satisfying (7) exists, we consider the inverse marginal functions

$$I_0 = (U'_0)^{(-1)} \quad \text{and} \quad I_1 = (U'_1)^{(-1)}.$$

Our main goal in this section is to show that the inverse investment problem (7) reduces to a functional equation in terms of I_0 and I_1 , see (9) below.

Next, we provide one of the main results herein, establishing a direct relationship between the inverse marginals at the beginning and the end of the trading period $[0, 1]$, when the corresponding utilities are related by (7).

Theorem 3. *Let $U_0, U_1 \in \mathcal{U}$ satisfy (7). Then, their inverse marginals I_0 and I_1 must satisfy the linear functional equation*

$$I_1(ay) + bI_1(y) = (1+b)I_0(cy); \quad y > 0, \quad (9)$$

where

$$a = \frac{1-p}{p} \frac{q}{1-q}, \quad b = \frac{1-q}{q} \quad \text{and} \quad c = \frac{1-p}{1-q}. \quad (10)$$

Proof. From standard arguments, an optimizer $\pi^*(x)$ exists for (7). Therefore,

$$pU_1(x + \pi^*(x)(u-1)) + (1-p)U_1(x + \pi^*(x)(d-1)) = U_0(x).$$

Differentiating the above and using the envelope condition

$$p(u-1)U'_1(x + \pi^*(x)(u-1)) + (1-p)U'_1(x + \pi^*(x)(d-1)) = 0, \quad (11)$$

yields

$$pU'_1(x + \pi^*(x)(u-1)) + (1-p)U'_1(x + \pi^*(x)(d-1)) = U'_0(x). \quad (12)$$

Solving the linear system (11)-(12) gives

$$U'_1(x + \pi^*(x)(u-1)) = \frac{(1-d)}{p(u-d)}U'_0(x)$$

and

$$U'_1(x + \pi^*(x)(d-1)) = \frac{(u-1)}{(1-p)(u-d)}U'_0(x).$$

Therefore, the optimal allocation function satisfies

$$\begin{aligned} x + \pi^*(x)(u-1) &= I_1 \left(\frac{1-d}{p(u-d)}U'_0(x) \right), \\ x + \pi^*(x)(d-1) &= I_1 \left(\frac{u-1}{(1-p)(u-d)}U'_0(x) \right), \end{aligned} \quad (13)$$

respectively. We obtain the solution

$$\pi^*(x) = \frac{1}{u-d} \left(I_1 \left(\frac{1-d}{p(u-d)} U'_0(x) \right) - I_1 \left(\frac{u-1}{(1-p)(u-d)} U'_0(x) \right) \right),$$

and, in turn, substituting in either of equations in (13) yields

$$\frac{1-d}{u-d} I_1 \left(\frac{1-d}{p(u-d)} U'_0(x) \right) + \frac{u-1}{u-d} I_1 \left(\frac{u-1}{(1-p)(u-d)} U'_0(x) \right) = x.$$

Using the change of variables $x = I_0 \left(\frac{(1-p)(u-d)}{u-1} y \right)$, $y > 0$, the above becomes

$$I_1 \left(\frac{(1-p)(1-d)}{p(u-1)} y \right) + \frac{u-1}{1-d} I_1(y) = \frac{u-d}{1-d} I_0 \left(\frac{(1-p)(u-d)}{u-1} y \right),$$

and recalling (10) we conclude. \square

Next, we show by an explicit construction that the problem of identifying U_1 from U_0 reduces to solving the functional equation (9), with I_0 given and I_1 to be found. We also derive the optimal portfolio $\pi^*(x)$ and its wealth $X^*(x)$.

Theorem 4. *Let U_0 be a utility function and I_0 be its inverse marginal. Furthermore, let I_1 be an inverse marginal satisfying the functional equation (9). The following statements hold.*

i) *The function U_1 defined by*

$$U_1(x) = U_0(1) + E_{\mathbb{P}} \left[\int_{I_1(\rho_1 U'_0(1))}^x I_1^{(-1)}(\xi) d\xi \right]; \quad x > 0, \quad (14)$$

is a well-defined utility function.

ii) *We have $U_0(x) = \sup_{\pi \in \mathcal{A}(x)} E_{\mathbb{P}} \left[U_1 \left(x + \pi(R-1) \right) \right]$, for all $x > 0$.*

iii) *The optimal wealth $X_1^*(x)$ and the associated optimal investment allocation $\pi^*(x)$ are given, respectively, by*

$$\begin{aligned} X^*(x) &= I_1(\rho_1 U'_0(x)) = X^{*,u}(x) \mathbf{1}_{\{R=u\}} + X^{*,d}(x) \mathbf{1}_{\{R=d\}}, \\ \pi^*(x) &= \frac{X^{*,u}(x) - X^{*,d}(x)}{u-d}, \end{aligned}$$

with

$$X^{*,u} = I_1 \left(\frac{q}{p} U'_0(x) \right) \quad \text{and} \quad X^{*,d} = I_1 \left(\frac{1-q}{1-p} U'_0(x) \right).$$

Proof. See Appendix A. \square

Remark 5. *As shown in the proof of Theorem 4, we can replace (14) with*

$$U_1(x) = U_0(c) + E_{\mathbb{P}} \left[\int_{I_1(\rho_1 U'_0(c))}^x I_1^{(-1)}(\xi) d\xi \right]; \quad x > 0,$$

for any arbitrary value $c > 0$. The choice of c does not change the value of $U_1(x)$.

Theorem 4 reduces the inverse investment problem (7) to the functional equation (9). We discuss this functional equation in the next section.

6 A functional equation for inverse marginals

In this section, we analyze linear functional equations of form

$$F(ay) + bF(y) = (1 + b)G(cy); \quad y > 0, \quad (15)$$

where G is a given function, $a, b, c > 0$ and F is to be found. We provide conditions for the existence and uniqueness of its solutions and, in particular, solutions in the class of inverse marginal functions. We assume that $a \neq 1$, since for $a = 1$, the unique solution is $F(y) = G(cy)$.

Equation (15) is a functional equation² of the form

$$F(f(y)) = g(y)F(y) + h(y),$$

with f, g , and h given functions, $y \in \mathcal{Y} \subseteq \mathbb{R}$ and F to be found. In general, such equations have “too many” solutions. A trivial example is $F(y+1) = F(y)$, $y \in \mathbb{R}$, for which any periodic function with period 1 is a solution. Such non-uniqueness often renders the underlying equation inapplicable for concrete problems, where a single well-defined solution is usually needed. Therefore, we first need to identify conditions for the uniqueness to hold. Such conditions usually limit the set of solutions by imposing additional assumption on $F(y_0)$, where y_0 is a fixed point for f , $f(y_0) = y_0$. In the aforementioned example, if we require a solution to be such that $\lim_{y \uparrow \infty} F(y)$ exists, then $F \equiv 0$ becomes the only viable solution. Note here that ∞ is a fixed point of the function $f(y) = y + 1$.

As the following example shows, the solution of (15) may not be unique, even if we restrict the solutions to functions that are inverse marginal.

Example 6. Let $\log_a b < 0$ and $G(y) = y^{\log_a b}$, $y > 0$. It is easy to check that the function $F_1(y) = \delta y^{\log_a b}$, $y > 0$, with $\delta = \frac{(1+b)}{2b c^{-\log_a b}} > 0$, is a solution. However, this particular solution is not the only one satisfying (9). Indeed, consider any differentiable anti-periodic function, say $\Theta(z) = -\Theta(z + \ln a)$, for which there exists a constant $M > 0$ such that

$$\sup_{z \in \mathbb{R}} (|\Theta(z)|, |\Theta'(z)|) < M < \delta \frac{-\log_a b}{1 - \log_a b} = -\frac{(1+b) \log_a b}{2b c^{-\log_a b} (1 - \log_a b)}.$$

For instance, let $\Theta(x) = M \sin(\frac{x}{\ln a} \pi)$. One can then directly check that the function

$$F_2(y) = y^{\log_a b} (\delta + \Theta(\ln y)),$$

$y > 0$, is a solution.

We remark that even in the class of inverse marginal functions we are interested herein, the uniqueness fails. Indeed, note that both solutions F_1 and F_2 are inverse marginals. This is obvious for F_1 . To establish it for F_2 , we first observe that it satisfies the Inada conditions. Indeed, $\lim_{y \uparrow \infty} F_2(y) = 0$ since $\log_a b < 0$ and $\lim_{y \downarrow 0} F_2(y) = \infty$, as it follows from the inequality $F_2(y) \geq y^{\log_a b} (\delta - M)$, $y > 0$. Moreover, F_2 is strictly decreasing since, for $y > 0$,

$$\begin{aligned} F_2'(y) &= y^{\log_a b - 1} \log_a b \left(\delta + \Theta(\ln y) + \frac{\Theta'(\ln y)}{\log_a b} \right) \\ &\leq y^{\log_a b - 1} \log_a b \left(\delta - \frac{M \log_a b - M}{\log_a b} \right) < 0. \end{aligned}$$

□

²See [6] and references therein for a general exposition.

For equation (15), $f(y) = ay$, $g(y) = -b$ and $h(y) = (1+b)G(cy)$. Therefore, uniqueness conditions should impose additional assumptions on F at $y_1 = 0$ and $y_2 = \infty$, which are the fixed points of $f(y) = ay$. We start with the following auxiliary result, which provide such uniqueness conditions. In particular, note that function F_2 in the previous example does not satisfy either conditions in Lemma 7.

Lemma 7. *Let G be given in equation (15). Then, there is at most one solution, say F , that satisfies $\lim_{y \downarrow 0} y^{-\log_a b} F(y) = 0$ and at most one solution that satisfies $\lim_{y \uparrow \infty} y^{-\log_a b} F(y) = 0$.*

Proof. Let F_1 and F_2 be two solutions of (9) that both satisfy either condition (i) or (ii). We show that their difference $w = F_1 - F_2 \equiv 0$.

The function w satisfies the homogenous equation $w(ay) = -bw(y)$, $y > 0$. Therefore, for $k \in \{1, 2, \dots\}$,

$$w(y) = \frac{w(ay)}{-b} = \frac{w(a^2y)}{(-b)^2} = \dots = \frac{w(a^ky)}{(-b)^k},$$

and

$$w(y) = -bw\left(\frac{y}{a}\right) = (-b)^2w\left(\frac{y}{a^2}\right) = \dots = (-b)^kw\left(\frac{y}{a^k}\right).$$

It then follows that for $k \in \{\pm 1, \pm 2, \dots\}$ and $y > 0$,

$$\begin{aligned} |w(y)| &= b^k \left| w\left(\frac{y}{a^k}\right) \right| = y^{\log_a b} \left(\frac{y}{a^k}\right)^{-\log_a b} \left| w\left(\frac{y}{a^k}\right) \right| \\ &\leq y^{\log_a b} \left(\frac{y}{a^k}\right)^{-\log_a b} \left(\left| F_1\left(\frac{y}{a^k}\right) \right| + \left| F_2\left(\frac{y}{a^k}\right) \right| \right). \end{aligned}$$

The right side vanishes as either $k \uparrow \infty$ or $k \downarrow -\infty$, and we easily conclude. \square

Next, we state our main result for this section, which provides sufficient conditions for existence and uniqueness of solutions to (15) that are inverse marginal functions.

Theorem 8. *Let G in (15) be an inverse marginal utility, i.e. $G \in \mathcal{I}$ with \mathcal{I} as in (8). Define, for $y > 0$, the functions*

$$\Phi_0(y) = G(acy) - bG(cy) \quad \text{and} \quad \Psi_0(y) = y^{-\log_a b} G(cy). \quad (16)$$

The following assertions hold:

- i) If Φ_0 is strictly increasing and, either $a > 1$ and $\lim_{y \uparrow \infty} \Psi_0(y) = 0$ or $a < 1$ and $\lim_{y \downarrow 0} \Psi_0(y) = 0$, then a solution of (15) is given, for $y > 0$, by*

$$F(y) = \frac{1+b}{b} \sum_{m=0}^{\infty} (-1)^m b^{-m} G(a^m cy). \quad (17)$$

- ii) If Φ_0 is strictly decreasing and, either $a > 1$ and $\lim_{y \downarrow 0} \Psi_0(y) = 0$ or $a < 1$ and $\lim_{y \uparrow \infty} \Psi_0(y) = 0$, then a solution of (15) is given, for $y > 0$, by*

$$F(y) = (1+b) \sum_{m=0}^{\infty} (-1)^m b^m G(a^{-(m+1)} cy).$$

- iii) In parts (i) and (ii), the corresponding F satisfies the uniqueness condition(s) of Lemma 4 and, moreover, $F \in \mathcal{I}$.
- iv) The function F in parts (i) and (ii), respectively, is the only positive solution of equation (15). Furthermore, F is the only inverse marginal that solves (15).

Proof. See Appendix B. □

Next, we apply the above result to an initial power utility. The following example provides results complementary to the ones in Example 6 in which uniqueness lacks as a result of conditions of Lemma 7 not holding.

Corollary 9. *Let, $U_0(x) = \frac{x^{1-\frac{1}{\theta}}}{1-1/\theta}$, for $x > 0$, and assume that $\theta \neq -\log_a b$, with $a, b, c > 0$ given by (10). Then, one has:*

- i) *The unique marginal utility function that satisfies the functional equation (9) with the initial data $I_0(y) = y^{-\theta}$ is given, for $y > 0$, by*

$$I_1(y) = \delta y^{-\theta} \quad \text{where} \quad \delta = \frac{1+b}{c^\theta(a^{-\theta}+b)}. \quad (18)$$

- ii) *The unique utility function U_1 that satisfies the inverse investment problem 7 is given, for $x \geq 0$, by*

$$U_1(x) = \frac{\delta^{1/\theta} x^{1-1/\theta}}{1-1/\theta}$$

and the corresponding optimal allocation by

$$\pi^*(x) = \frac{x}{u-1} \left(\delta \left(\frac{p}{q} \right)^\theta - 1 \right).$$

Proof. Assertion (ii) follows from (i) and Theorem 4. Also, one can easily check that I_1 given by (18) is an inverse marginal and satisfies (9).

It only remains to show the uniqueness of solutions that are inverse marginals. To this end, it suffices to check that the condition of Theorem 8 holds for all possible values of the parameters. Setting $G(y) = y^{-\theta}$, $y > 0$, in (16) yields

$$\Phi_0(y) = (a^{-\theta} - b)c^{-\theta}y^{-\theta} \quad \text{and} \quad \Psi_0(y) = y^{-(\theta+\log_a b)}.$$

Since $\theta \neq -\log_a b$ and $a \neq 1$, we have the following dichotomy:

- a) Either $\theta < -\log_a b$ and $a < 1$ or $\theta > -\log_a b$ and $a > 1$. Then, one can show that conditions (i) of Theorem 8 holds.
- b) Either $\theta < -\log_a b$ and $a > 1$ or $\theta > -\log_a b$ and $a < 1$. Then, one can show that conditions (ii) of Theorem 8 holds. □

We end this section by providing the main result concerning the solution to the inverse Merton problem, and then utilize it to explicitly state the steps needed to construct predictable forward performance processes as we explained in Section 4.

In the following theorem, let a , b , and c be as in (10) and, for $I_0 = (U'_0)^{(-1)} \in \mathcal{I}$, define

$$\Phi_0(y) = I_0(a c y) - b I_0(c y) \quad \text{and} \quad \Psi_0(y) = y^{-\log_a b} I_0(c y),$$

for $y > 0$. Furthermore, consider the following cases:

C1: Φ_0 is strictly increasing and, either $a > 1$ and $\lim_{y \uparrow \infty} \Psi_0(y) = 0$ or $a < 1$ and $\lim_{y \downarrow 0} \Psi_0(y) = 0$. In this case define, for $y > 0$,

$$I_1(y) = \frac{1+b}{b} \sum_{m=0}^{\infty} (-1)^m b^{-m} I_0(a^m c y). \quad (19)$$

C2: Φ_0 is strictly decreasing and, either $a > 1$ and $\lim_{y \downarrow 0} \Psi_0(y) = 0$ or $a < 1$ and $\lim_{y \uparrow \infty} \Psi_0(y) = 0$. In this case define, for $y > 0$,

$$I_1(y) = (1+b) \sum_{m=0}^{\infty} (-1)^m b^m I_0(a^{-(m+1)} c y). \quad (20)$$

Theorem 10. *Consider the inverse Merton problem (7), assume that the initial inverse marginal $I_0 = (U'_0)^{(-1)}$ satisfies condition C1 (resp. condition C2) above, and define I_1 by (19) (resp. (20)). Then, the unique solution to (7) is given by*

$$U_1(x) = U_0(1) + E_{\mathbb{P}} \left[\int_{I_1(\rho_1 U'_0(1))}^x I_1^{(-1)}(\xi) d\xi \right]; \quad x > 0,$$

where, ρ_1 is given by (6). The optimal wealth $X_1^*(x)$ and the associated optimal investment allocation $\pi^*(x)$ are given, respectively, by

$$\begin{aligned} X^*(x) &= I_1(\rho_1 U'_0(x)) = X^{*,u}(x) \mathbf{1}_{\{R=u\}} + X^{*,d}(x) \mathbf{1}_{\{R=d\}}, \\ \pi^*(x) &= \frac{X^{*,u}(x) - X^{*,d}(x)}{u - d}, \end{aligned}$$

with

$$X^{*,u} = I_1 \left(\frac{q}{p} U'_0(x) \right) \quad \text{and} \quad X^{*,d} = I_1 \left(\frac{1-q}{1-p} U'_0(x) \right).$$

Proof. Under conditions C1 or C2, Theorem 8 yields that I_1 is the unique solution of (9). U_1 , X^* and π^* are then obtained by Theorem 4. \square

Going back to Section 4, we may now explicitly state the steps needed to construct (and invest according to) predictable forward performance processes. Assume that at $t = 0$, we have initial wealth X_0 , and that we know our initial utility function U_0 . Let $I_0 = (U')^{-1}$. We then iterate the following step for $n \in \{0, 1, \dots\}$.

At $t = t_n$: We observe the parameters for the next period $[t_n, t_{n+1}]$, namely $(R_{n+1}^u, R_{n+1}^d, p_{n+1})$. If $n \geq 1$, we also observe R_n , the asset return over the previous period $[t_{n-1}, t_n]$, and obtain our current wealth $X_n = X_{n-1} + \pi_n(R_n - 1)$. Let

$$q_{n+1} = \frac{1 - R_{n+1}^d}{R_{n+1}^u - R_{n+1}^d}, \quad a = \frac{q_{n+1}(1 - p_{n+1})}{p_{n+1}(1 - q_{n+1})}, \quad b = \frac{1 - q_{n+1}}{q_{n+1}}, \quad \text{and} \quad c = \frac{1 - p_{n+1}}{1 - q_{n+1}}.$$

Check conditions C1 and C2 for $I_0 = I_n$. If condition C1 (resp. C2) hold, define I_{n+1} by the right side of (19) (resp. (20)). If both conditions fail, then see Remark 11 below.

Then, for period period $[t_n, t_{n+1}]$, the utility function is given by

$$\begin{aligned} U_{n+1}(x) = U_n(1) + p_{n+1} \int_{I_{n+1}(\rho_{n+1}^u U'_n(1))}^x I_{n+1}^{(-1)}(\xi) d\xi \\ + (1 - p_{n+1}) \int_{I_{n+1}(\rho_{n+1}^d U'_n(1))}^x I_{n+1}^{(-1)}(\xi) d\xi; \quad x > 0, \end{aligned}$$

and the optimal investment is given by

$$\pi_{n+1} = \frac{1}{R_{n+1}^u - R_{n+1}^d} [I_{n+1}(\rho_{n+1}^u U'_n(X_n)) - I_{n+1}(\rho_{n+1}^d U'_n(X_n))],$$

where

$$\rho_{n+1}^u = \frac{q_{n+1}}{p_{n+1}} \quad \text{and} \quad \rho_{n+1}^d = \frac{1 - q_{n+1}}{1 - p_{n+1}}.$$

Remark 11. *If both conditions C1 and C2 do not hold, then the functional equation (9) may not have a solution, or the solution may not be unique. For the case of power utility $U_0(x) = \frac{x^{1-1/\theta}}{1-1/\theta}$, $\theta > 0$, Example 6 and Corollary 9 yield that both condition fail at t_n if and only if $\theta = -\log_a b > 0$, in which case, the solution exists but not unique. We leave the case when both condition C1 and C2 are violated as a future research direction.*

7 Conclusion

We proposed a novel approach to optimal portfolio choice that allows for investor preferences and model specification to be known only for a short time ahead. This approach is a hybrid of the classical expected utility approach and the forward performance processes introduced by [9] and [10], in the sense that the risk preferences are stochastic and updated by the forward approach at the end of each period while, within each period, the investor faces a classical expected utility maximization problem. Since investor's preference is know for a short time ahead, we call our criterion a predictable forward performance process.

We considered a binomial model with random parameters which accommodates short-term predictability of asset returns, sequential learning and other dynamically unfolding factors affecting optimal portfolio choice. We then discussed in detail how the construction of predictable forward performance processes reduces to an inverse investment problem.

This inverse investment problem is, in turn, shown to be equivalent to a functional equation in terms of the inverse marginal functions. We established conditions for existence and uniqueness of solutions in the class of inverse marginal functions. This functional equation is the counterpart of the stochastic partial differential equation that characterizes the continuous-time forward performance processes.

References

- [1] O.E. Barndorff-Nielsen and N. Shephard. Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 64(2):253–280, 2002.
- [2] F. Black. Individual investment and consumption under uncertainty. In D. L. Luskin, editor, *Portfolio Insurance: A Guide to Dynamic Hedging*, pages 207–225. John Wiley and Sons, New York, 1988.
- [3] N. El Karoui and M. Mrad. An exact connection between two solvable SDEs and a nonlinear utility stochastic PDE. *SIAM Journal on Financial Mathematics*, 4(1):697–736, 2013.
- [4] E. F. Fama and K. R. French. Business conditions and expected returns on stocks and bonds. *Journal of financial economics*, 25(1):23–49, 1989.
- [5] S. Kallblad, J. Obłój, and T. Zariphopoulou. Time-consistent investment under model uncertainty: the robust forward criteria. Working paper, Available at arXiv:1311.3529, 2013.
- [6] M. Kuczma, B. Choczewski, and R. Ger. *Iterative Functional Equations*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1990.
- [7] T. Leung, R. Sircar, and T. Zariphopoulou. Forward indifference valuation of american options. *Stochastics An International Journal of Probability and Stochastic Processes*, 84(5-6):741–770, 2012.
- [8] M. Musiela, E. Sokolova, and T. Zariphopoulou. forward exponential indifference valuation in an incomplete binomial model. Working paper, 2016.
- [9] M. Musiela and T. Zariphopoulou. Backward and forward utilities and the associated pricing systems: The case study of the binomial model. In R. Carmona, editor, *Indifference Pricing*. Princeton University Press, 2003.
- [10] M. Musiela and T. Zariphopoulou. Portfolio choice under dynamic investment performance criteria. *Quantitative Finance*, 9(2):161–170, 2009.
- [11] M. Musiela and T. Zariphopoulou. Stochastic partial differential equations in portfolio choice. In C. Chiarella and A. Novikov, editors, *Contemporary Quantitative Finance*, pages 195–216. Springer-Verlag Berlin Heidelberg, 2010.
- [12] M. Musiela and T. Zariphopoulou. Initial investment choice and optimal future allocations under time-monotone performance criteria. *International Journal of Theoretical and Applied Finance*, 14(01):61–81, 2011.
- [13] S. Nadtochiy and M. Tehranchi. Optimal investment for all time horizons and martin boundary of space-time diffusions. *Mathematical Finance*, 2015.
- [14] S. Nadtochiy and T. Zariphopoulou. A class of homothetic forward investment performance processes with non-zero volatility. In *Inspired by Finance*, pages 475–504. Springer, 2014.

- [15] M. Shkolnikov, R. Sircar, and T. Zariphopoulou. Asymptotic analysis of forward performance processes in incomplete markets and their ill-posed hjb equations. Working paper, Available at arXiv:1504.03209, 2015.
- [16] T. Zariphopoulou and G. Zitkovic. Maturity-independent risk measures. *SIAM Journal on Financial Mathematics*, 1(1):266–288, 2010.
- [17] G. Žitković. A dual characterization of self-generation and exponential forward performances. *The Annals of Applied Probability*, 19(6):2176–2210, 2009.

A Proof of Theorem 4

We need the following auxiliary result, showing that (7) is equivalent to

$$U_0(I_0(y)) = E_{\mathbb{P}} \left[U_1(I_1(\rho_1 y)) \right]; \quad y > 0. \quad (21)$$

Lemma 12. *Suppose that $U_0, U_1 \in \mathcal{U}$ and let I_0 and I_1 be their inverse marginals. Then, (7) holds if and only if (21) holds.*

Proof. (7) \Rightarrow (21): Standard results yield that (7) implies

$$U_0(x) = E_{\mathbb{P}} \left[U_1(I_1(\rho_1 U'_0(x))) \right]; \quad x > 0,$$

and (21) is obtained by the change of variable $y = U'_0(x)$.

(21) \Rightarrow (7): Assume that (21) holds and define the value function \tilde{U}

$$\tilde{U}(x) = \sup_{\mathcal{A}(x)} E_{\mathbb{P}} [U_1(X)]; \quad x > 0.$$

We will show that $\tilde{U} \equiv U_0$. Let \tilde{I} be the inverse marginal of \tilde{U} . By (i), one must have

$$\tilde{U}(\tilde{I}(y)) = E_{\mathbb{P}} \left[U_1(I_1(\rho_1 y)) \right]; \quad y > 0,$$

and it follows that $\tilde{U}(\tilde{I}(y)) = U_0(I_0(y))$ for $y > 0$. Differentiating with respect to y yields $\tilde{I}' \equiv I'_0$. Therefore $\tilde{I}(y) = I_0(y) + C$, $y > 0$, for some constant C . Taking the limit as $y \rightarrow +\infty$ and using the Inada condition $\tilde{I}(+\infty) = I_0(+\infty) = 0$ yields that $C = 0$. Therefore, we obtain $\tilde{I} \equiv I_0$, which implies $\tilde{U}'(x) = U'_0(x)$, for all $x > 0$. Finally, we obtain

$$\tilde{U}(x) = E_{\mathbb{P}} \left[U_1(I_1(\rho \tilde{U}'(x))) \right] = E_{\mathbb{P}} \left[U_1(I_1(\rho U'_0(x))) \right] = U_0(x); \quad x > 0. \quad \square$$

Proof of Theorem 4. (i): By (14),

$$U_1(x) := U_0(1) + p \times \int_{x_u(1)}^x I_1^{(-1)}(\xi) d\xi + (1-p) \times \int_{x_d(1)}^x I_1^{(-1)}(\xi) d\xi; \quad x > 0,$$

where x_u and x_d as given by

$$x_i(c) = I_1(\rho^i U'_0(c)); \quad c > 0, i \in \{u, d\}. \quad (22)$$

Thus,

$$U_1'(x) = p I_1^{(-1)}(x) + (1-p) I_1^{(-1)}(x) = I_1^{(-1)}(x); \quad x > 0.$$

It then follows that I_1 is the inverse marginal of U_1 and that U_1 is a utility function.

(ii): Define the function F

$$F(x, c) := U_0(c) + p \int_{x_u(c)}^x I_1^{(-1)}(\xi) d\xi + (1-p) \int_{x_d(c)}^x I_1^{(-1)}(\xi) d\xi; \quad (x, c) \in \mathbb{R}^+ \times \mathbb{R}^+, \quad (23)$$

where x_u and x_d are given by (22). We show that

$$\frac{\partial F}{\partial c}(x, c) = 0; \quad \forall x, c \in \mathbb{R}^+.$$

Differentiating (23) with respect to c and then using $I_1^{-1}(x_i(c)) = \rho^i U_0'(c)$, for $c > 0$, yields

$$\begin{aligned} \frac{\partial F}{\partial c}(x, c) &= U_0'(c) - p x_u'(c) G(x_u(c)) - (1-p) x_d'(c) G(x_d(c)) \\ &= U_0'(c) - p x_u'(c) \rho^u U_0'(c) - (1-p) x_d'(c) \rho^d U_0'(c) \\ &= U_0'(c) \left\{ 1 - p \rho^u x_u'(c) - (1-p) \rho^d x_d'(c) \right\} = 0. \end{aligned}$$

To obtain the last equation, note that (9) is equivalent to

$$I_0(y) = p \rho^u I_1(y \rho_u) + (1-p) \rho^d I_1(y \rho_d); \quad y > 0.$$

Therefore, substituting $y = U_0(c)$ and differentiating with respect to c yields,

$$\begin{aligned} 1 &= \frac{d}{dc} \left(I_0'(U_0(c)) \right) = \frac{d}{dc} \left[p \rho^u I_1'(\rho_u U_0'(c)) + (1-p) \rho^d I_1'(\rho_d U_0'(c)) \right] \\ &= p (\rho^u)^2 I_1'(\rho_u U_0'(c)) U_0''(c) + p (\rho^d)^2 I_1'(\rho_d U_0'(c)) U_0''(c) \\ &= p \rho^u x_u'(c) + (1-p) \rho^d x_d'(c). \end{aligned}$$

Note that, by definition, $U_1(x) = F(x, 1)$. Since we showed $\frac{\partial F}{\partial c} \equiv 0$, we must have $U_1(x) = F(x, c)$ for all $x > 0$ and $c > 0$. In other words, for all $x, c \in \mathbb{R}^+$, U_1 satisfies

$$U_1(x) = U_0(c) + p \int_{x_u(c)}^x I_1^{(-1)}(\xi) d\xi + (1-p) \int_{x_d(c)}^x I_1^{(-1)}(\xi) d\xi$$

As shown in (i), $U_1' \equiv I_1^{(-1)}$. Therefore, for all $x > 0$ and $c > 0$,

$$U_1(x) = U_0(c) + p \left[U_1(x) - U_1(x_u(c)) \right] + (1-p) \left[U_1(x) - U_1(x_d(c)) \right],$$

which, in turn, yields

$$U_0(c) = p U_1(x_u(c)) + (1-p) U_1(x_d(c)) = E_{\mathbb{P}} \left[U_1(I_1(\rho_1 U_0'(c))) \right]; \quad \forall c > 0.$$

This is equivalent to (21). Hence, (ii) follows by Lemma 12.

(iii): This part easily follows from existing results in traditional expected utility problems if we view (7) as a terminal expected utility problem with U_1 now given and U_0 being its value function. \square

B Proof of Theorem 8

We only show part (i) and the corresponding statements in parts (iii) and (iv), since (ii) follows along similar arguments.

(i) Direct substitution shows that if the infinite series in (17) converges, then F satisfies (15). Thus, to show (i), it only remains to show that under the aforementioned conditions, the series converges. Note that (17) can be written, for $y > 0$, as

$$F(y) = \frac{b}{1+b} y^{\log_a b} \sum_{m=0}^{\infty} (-1)^m \Psi_0(a^m y), \quad (24)$$

which, by the Leibniz test for alternating series, converges if $\lim_{m \rightarrow \infty} \Psi_0(a^m y) = 0$ monotonically. $\lim_{m \rightarrow \infty} \Psi_0(a^m y) = 0$ follows directly from either of conditions in (i) on a and Ψ_0 . To show that the convergence is monotonic, note that by (16)

$$\Psi_0(a^{m+1} y) - \Psi_0(a^m y) = b^{-m-1} y^{-\log_a b} \Phi_0(a^m y); \quad y > 0, \quad m \in \{0, 1, \dots\}. \quad (25)$$

On the other hand, since Φ_0 is increasing and $\lim_{y \uparrow \infty} \Phi_0(y) = \lim_{y \uparrow \infty} G(a c y) - b G(c y) = 0$ by the Inada condition, we must have $\Phi_0(y) < 0$ for $y > 0$. Thus, by (25), $\Psi_0(a^m y) > \Psi_0(a^{m+1} y)$, and $\lim_{m \uparrow \infty} \Psi_0(a^m y) = 0$ monotonically.

(iii) First, observe that F is strictly decreasing. Indeed, (24) and (25) yield

$$F(y) = \frac{b}{1+b} y^{\log_a b} \sum_{m=0}^{\infty} \left(\Psi_0(a^{2m} y) - \Psi_0(a^{2m+1} y) \right) = -\frac{1}{1+b} \sum_{m=0}^{\infty} b^{-2m} \Phi_0(a^{2m} y).$$

It then follows that, for $y < y'$,

$$F(y') - F(y) = \frac{1}{1+b} \sum_{m=0}^{\infty} b^{-2m} \left(\Phi_0(a^{2m} y) - \Phi_0(a^{2m} y') \right) < 0,$$

where the inequality holds because Φ_0 is strictly increasing.

Using equation (15), that $a, b, c > 0$ and $\lim_{y \uparrow \infty} G(y) = 0$, and the monotonicity of F , we deduce that $\lim_{y \uparrow \infty} F(y) = 0$, and, in turn, that $F(y) > 0$, $y > 0$. Similarly, $\lim_{y \downarrow 0} G(y) = \infty$, yields $\lim_{y \downarrow 0} F(y) = \infty$. Thus, we have shown that $F \in \mathcal{I}$.

Finally, conditions in Lemma 7 follow from $\Psi_0(y) \rightarrow 0$ as either $y \downarrow 0$ or $y \uparrow \infty$ and that

$$0 < y^{\log_a b} F(y) = \frac{F(y)}{G(c y)} \Psi_0(y) < \frac{b+1}{b} \Psi_0(y); \quad \forall y > 0,$$

where we used (15) and that $F(y) > 0$ to obtain

$$\frac{F(y)}{G(c y)} = \frac{(1+b)F(y)}{F(a y) + b F(y)} < \frac{1+b}{b}.$$

(iv) Repeating the last part of the argument in part (iii) for any solution $\tilde{F} > 0$, yields that \tilde{F} satisfies the same uniqueness condition (15) as F . The result immediately follows from Lemma 7.